

# A surprise in mechanics with nonlinear chiral supermultiplet

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## Abstract

We show that the nonlinear chiral supermultiplet allows one to construct, over given two-dimensional bosonic mechanics, the family of two-dimensional  $\mathcal{N} = 4$  supersymmetric mechanics parameterized with the holomorphic function  $\lambda(z)$ . We show, that this family includes, as a particular case, the  $\mathcal{N} = 4$  superextensions of two-dimensional mechanics with magnetic fields, which have factorizable Schroedinger equations.

## Introduction

Since its discovery [1] supersymmetric mechanics attracts much interest as a convenient toy model for the study of dynamical consequences of supersymmetry. It is also a convenient object for developing the supersymmetry technique, particularly for the construction of supersymmetric models within the superfield approach. However, even in the latter case supersymmetric mechanics was found to have some specific properties, which have no analogs in dimensions higher than one. For instance, in [2] it was found, that in  $\mathcal{N} = 4, d = 1$  supersymmetry, besides the five off-shell linear finite supermultiplets [3] and the one-dimensional analog of the  $\mathcal{N} = 2, d = 4$  nonlinear multiplet [4], there exists some new nonlinear supermultiplet, (called in [2] nonlinear chiral supermultiplet) which seems to have no known higher-dimensional analogs. It includes, as a limiting case, the standard chiral supermultiplet and has the same components as the latter. Let us recall that the standard (linear) chiral supermultiplet corresponds to the complex superfield parameterizing the two-dimensional plane  $\mathbb{R}^2 = \mathbb{C}^1$ . Opposite to that case, the nonlinear chiral supermultiplet corresponds to the complex superfield parameterizing the two-dimensional sphere (complex projective plane)  $S^2 = \mathbb{CP}^1 = \text{SU}(2)/\text{U}(1)$ , but it has the same component content, as the linear one. Consequently, the standard chirality condition is modified as follows:

$$D_i \mathcal{Z} = -\alpha \mathcal{Z} \bar{D}_i \mathcal{Z}, \quad \bar{D}_i \bar{\mathcal{Z}} = \alpha \bar{\mathcal{Z}} D_i \bar{\mathcal{Z}}, \quad \alpha = \text{const.} \quad (1)$$

In Ref.[5], by the use of the nonlinear chiral supermultiplet, the model of two-dimensional  $\mathcal{N} = 4$  supersymmetric mechanics has been suggested, with the following superfield action:

$$S = \int dt d^2 \theta d^2 \bar{\theta} K(\mathcal{Z}, \bar{\mathcal{Z}}) + \int dt d^2 \bar{\theta} F(\mathcal{Z}) + \int dt d^2 \theta \bar{F}(\bar{\mathcal{Z}}). \quad (2)$$

Here  $K(\mathcal{Z}, \bar{\mathcal{Z}})$  is an arbitrary real function playing the role of Kähler potential of the metric, while  $F(\mathcal{Z})$  and  $\bar{F}(\bar{\mathcal{Z}})$  are arbitrary holomorphic and antiholomorphic functions. Some interesting features of the model were observed there, e.g. the possibility to incorporate a magnetic field preserving the supersymmetry of the system. It was shown that this system includes, as a particular case, the  $\mathcal{N} = 4$  supersymmetric Landau problem on the sphere. Later on the nonlinear chiral multiplet has been used for the construction of  $\mathcal{N} = 8$  supersymmetric mechanics [6], as well as obtained by the reduction of the linear supermultiplet with four bosonic and four fermionic degrees of freedom [7].

In the present note we show that  $\mathcal{N} = 4$  supersymmetric mechanics with nonlinear chiral multiplet possesses a quite surprising property.

When we construct the supersymmetric mechanics with linear chiral multiplet, the arbitrariness of the construction is in the choice of Kähler potential  $K$  and superpotential  $F(z)$  only, and these functions define the underlying bosonic configuration. The extension to a supersymmetric system is unique [8]. On the contrary, when dealing with nonlinear chiral supermultiplet, we have the freedom in the supersymmetric extension of the given bosonic system, encoded in the choice of the holomorphic function  $\lambda(z)$ . When the underlying bosonic system is of the sigma-model type, the function  $\lambda(z)$  remains arbitrary. Otherwise it is related with the given potential and magnetic field as follows:

$$U(z, \bar{z}) = \frac{F'(z) \bar{F}'(\bar{z})}{(1 + \lambda \bar{\lambda})^2 g}, \quad B = \frac{\bar{\lambda}'(\bar{z}) F'(z) + \lambda'(z) \bar{F}'(\bar{z})}{(1 + \lambda \bar{\lambda})^2 g}. \quad (3)$$

Here  $g dz d\bar{z}$  defines the Kähler metric of the underlying bosonic space,  $U$  is a potential of the underlying bosonic system, and  $B$  is the magnitude of the magnetic field.

An interesting feature of supersymmetric (quantum) mechanics is the application to integrable systems of quantum mechanics. Initially, it was found that supersymmetric quantum mechanics could be naturally related with the

factorisation method of the solution of one-dimensional Schroedinger equations, yielding the algebraic approach to the construction of the spectra of all known integrable one-dimensional quantum-mechanical systems [9]. Later on, the factorisation method, based on supersymmetric mechanics, has been applied to the specific higher-dimensional mechanics with spin (see, e.g. [10] and [11] for a review and references).

Recently, Ferapontov and Veselov performed a systematic study of the factorisation method of quantum mechanical systems on curved two-dimensional surfaces in the presence of magnetic field (without any use of the supersymmetry technique) [12]. In particular, they found the restrictions to the admissible set of potentials and magnetic fields, which allows for a factorisable Schroedinger equation. We shall show that for the specific choice  $\lambda = \pm F$ , the system under consideration yields a  $\mathcal{N} = 4$  superextension of Ferapontov-Veselov mechanics.

### $\lambda(z)$ -freedom

The action (2) of the  $\mathcal{N} = 4$  supersymmetric mechanics with nonlinear chiral supermultiplet can be represented as follows [5]:

$$S = \int dt \left\{ g \dot{z} \dot{\bar{z}} - i \bar{\lambda}(\bar{z}) \mathcal{F}_z \dot{z} + i \lambda(z) \bar{\mathcal{F}}_{\bar{z}} \dot{\bar{z}} - \frac{\mathcal{F}_z \bar{\mathcal{F}}_{\bar{z}}}{g} \right. \\ \left. + \frac{i}{4} h \left[ \psi^i D_t \bar{\psi}_i - (D_t \psi^i) \bar{\psi}_i + \frac{\dot{z} \lambda' \psi^2}{1 + \lambda \bar{\lambda}} + \frac{\dot{\bar{z}} \bar{\lambda}' \bar{\psi}^2}{1 + \lambda \bar{\lambda}} \right] - \frac{(1 + \lambda \bar{\lambda})}{4} \left[ \frac{h^2}{4} \left( \frac{\lambda' \bar{\lambda}'}{(1 + \lambda \bar{\lambda}) h} - \mathcal{R} \right) \psi^2 \bar{\psi}^2 + \mathcal{F}_{z;z} \psi^2 - \bar{\mathcal{F}}_{\bar{z};\bar{z}} \bar{\psi}^2 \right] \right\}. \quad (4)$$

The supercharges corresponding to the  $\mathcal{N} = 4$  supersymmetry transformations read

$$Q^i = \Theta^i - \bar{\lambda}(\bar{z}) \bar{\Theta}^i, \quad \bar{Q}_i = \bar{\Theta}_i + \lambda(z) \Theta_i, \quad (5)$$

where

$$\Theta^i = g \left( \dot{z} + \frac{i}{4} \bar{\lambda}'(\bar{z}) \bar{\psi}^2 \right) \psi^i + i \bar{\mathcal{F}}_{\bar{z}} \bar{\psi}^i, \quad \bar{\Theta}_i = g \left( \dot{\bar{z}} + \frac{i}{4} \lambda'(z) \psi^2 \right) \bar{\psi}_i + i \mathcal{F}_z \psi_i.$$

Here we introduced the following notation:

$$g(z, \bar{z}) = \partial \bar{\partial} K(z, \bar{z}), \quad \lambda(z) = \alpha z, \quad \bar{\lambda}(\bar{z}) = \alpha \bar{z}, \quad h(z, \bar{z}) = (1 + \lambda \bar{\lambda}) g, \quad \mathcal{F}_z = \frac{F'(z)}{1 + \lambda \bar{\lambda}}, \quad \bar{\mathcal{F}}_{\bar{z}} = \frac{\bar{F}'(\bar{z})}{1 + \lambda \bar{\lambda}} \quad (6)$$

and

$$D\psi^i = d\psi^i + \Gamma \psi^i dz, \quad \Gamma = \partial \log h, \quad \mathcal{R} = -\partial \bar{\partial} \log h / h, \quad \mathcal{F}_{z;z} = \partial \mathcal{F}_z - \Gamma \mathcal{F}_z. \quad (7)$$

It is clear that  $\Gamma$  and  $\mathcal{R}$  define, respectively, the connection and the scalar curvature of the metric  $h(z, \bar{z}) dz d\bar{z}$ , while  $\mathcal{F}_{z;z}$  is the covariant derivative of the one-form  $\mathcal{F}_z dz$  with respect to this metric.

The above expressions are not covariant with respect to holomorphic transformations  $z \rightarrow f(z)$ ,  $\psi \rightarrow f'(z) \psi$ . However, the covariance will be immediately restored, if we assume that  $\lambda(z)$  is an arbitrary holomorphic function, instead of  $\lambda = \alpha z$ , and  $\lambda'(z) = d\lambda(z)/dz$  instead of  $\lambda' = \alpha = \text{const}$ .

The key observation is that when  $F' = 0$ , the kinetic term of the underlying bosonic system does not change upon this replacement. *Hence, there are infinitely many ways to supersymmetrize a free particle (i.e. when  $F'(z) = 0$ ), since in this case the function  $\lambda(z)$  could be any!* This is a completely unexpected result: to our knowledge, supersymmetry was, in some sense, an occasional (or exceptional) property in mechanical systems with fermionic and bosonic degrees of freedom, being (almost completely) defined by the underlying bosonic configuration. Namely, for its appearance, a strong correlation between the spin interaction and the electric-magnetic one was needed. Instead, in the present model this is not the case.

The system contains the interaction with a *nonzero magnetic field* defined by the one-form  $\mathcal{A}_B$

$$\mathcal{A}_B = i \bar{\lambda}(\bar{z}) \mathcal{F}_z dz - i \lambda(z) \bar{\mathcal{F}}_{\bar{z}} d\bar{z}, \quad d\mathcal{A}_B = i \frac{\bar{\lambda}'(\bar{z}) \mathcal{F}_z + \lambda'(z) \bar{\mathcal{F}}_{\bar{z}}}{(1 + \lambda \bar{\lambda})^2} d\bar{z} \wedge dz. \quad (8)$$

Thus, the magnitude of the magnetic field and the bosonic potential are defined by the expression (3).

Hence, the appearance of the magnetic field and the potential yields a restriction in the freedom of choice of the  $\lambda(z)$  function, defining the coupling of the fermionic degrees of freedom; but even in this case it is not completely fixed.

## Ferapontov-Veselov systems

Let us consider the special case of the  $\mathcal{N} = 4$  supersymmetric mechanics with nonlinear chiral multiplet, when  $\lambda(z) = \pm F(z)$  (notice, that upon the choice  $\lambda = iF$  the potential term remains unchanged, but the magnetic field vanishes). Upon this choice one has

$$B = \pm 2U, \quad U = \frac{F'\bar{F}'}{(1 + F\bar{F})^2 g} . \quad (9)$$

Such systems possess a quite important property: it was shown by Ferapontov and Veselov, that they have a factorisable quantum Hamiltonian (Schroedinger operator) [12], closely related with supersymmetry [9, 10, 11]. Let us recall that the Schroedinger operator  $\hat{\mathcal{H}}_0$  is called factorisable, if it can be represented in the form  $\hat{\mathcal{H}}_0 = \hat{D}_1 \hat{D}_2$ , where  $\hat{D}_{1,2}$  are first-order differential operators. In this case the operator  $\hat{\mathcal{H}}_0 = \hat{D}_2 \hat{D}_1$  has the same spectrum as the former one, except, possibly, the state with zero energy! In the one-dimensional case one can calculate the complete spectra of such an operator in a purely algebraic way. Opposite to the one-dimensional case, the two-dimensional factorisable Schroedinger operators are quasi-exactly solvable, while only the operators with constant magnetic fields on the spaces with constant curvature admit complete exact solvability [12]. Also, in [12] it was found that if the two-dimensional Schroedinger equation with  $B = U = 0$  is integrable on some surface, then the Schroedinger equation with  $U = \pm B/2 = \mathcal{R}_0$  (where  $\mathcal{R}$  is the scalar curvature of the surface) is also integrable, and it has the same spectrum as the former one, except, possibly, the zero-energy level [12]. For the system under consideration this requirement yields the following restriction to the metrics:

$$g(z, \bar{z}) dz d\bar{z} = \frac{dz d\bar{z}}{(1 + \lambda\bar{\lambda})^2} . \quad (10)$$

In this case the scalar curvature of the metrics is given by the expression

$$\mathcal{R}_0 = \lambda'(z) \bar{\lambda}'(\bar{z}) . \quad (11)$$

Notice that upon this choice of the metric, the Lagrangian of the supersymmetric mechanics with nonlinear chiral multiplet looks much simpler, than with the generic one

$$\begin{aligned} \mathcal{L} = & \frac{\dot{z}\dot{\bar{z}}}{(1 + \lambda\bar{\lambda})^2} - i \frac{\bar{\lambda}(\bar{z})F'\dot{z} - \lambda(z)\bar{F}'\dot{\bar{z}}}{(1 + \lambda\bar{\lambda})} - F'\bar{F}' \\ & + \frac{i(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi})}{4(1 + \lambda\bar{\lambda})} + \frac{i(\dot{z}\lambda'\bar{\lambda} - \dot{\bar{z}}\bar{\lambda}'\lambda)\psi\bar{\psi} + \dot{z}\bar{\lambda}'\bar{\psi}^2 + \dot{\bar{z}}\lambda'\psi^2}{4(1 + \lambda\bar{\lambda})^2} - \frac{F''\psi^2 - \bar{F}''\bar{\psi}^2}{4} . \end{aligned} \quad (12)$$

Finally, let us notice that the factorisation method of the Schroedinger equation is generically with  $\mathcal{N} = 2$  supersymmetry, describing particles with spin 1/2. In the presented case we arrive, after quantization, to a spin 1 system. It seems clear that factorising our supersymmetric (quantum) Hamiltonian, we shall arrive to the pair of isospectral Hamiltonians for spin 1/2 systems.

## Conclusion

We have shown that the  $N = 4$  supersymmetric mechanics with the nonlinear chiral supermultiplet, constructed on  $S^2 = SU(2)/U(1)$  qualitatively differs from other supersymmetric mechanics models constructed within the superfield approach. The difference insists in the wide freedom in the supersymmetrization ways of the given bosonic system, encoded in the choice of the holomorphic function  $\lambda(z)$ . When the underlying bosonic system has no interaction with the external field this function remains arbitrary. Otherwise, it is restricted by the given form of potential and magnetic field. A particular choice of the  $\lambda(z)$ -function allows one to include in the class of  $\mathcal{N} = 4$  supersymmetrizable mechanics the two-dimensional systems with factorisable Schroedinger equation, analyzed by Ferapontov and Veselov. We believe that this simple example may drastically change the common intuitive impression about the rigidity of the supersymmetrization procedure.

Let us notice that by the use of chiral supermultiplet one can construct the  $\mathcal{N} = 8$  supersymmetric mechanics as well [2, 13]. The use of linear chiral multiplet yields the supersymmetric mechanics on special Kähler manifolds, with a strong restriction on the admissible set of potentials [13]. The  $\mathcal{N} = 8$  supersymmetric mechanics with nonlinear chiral supermultiplet [6] has a metrics with the deformed condition ensuring that the space be a special Kähler one. We are sure that, similarly to the above consideration, also in this case one can restore the  $\lambda(z)$  freedom, may be

with some additional restriction, as well as clarify the origin of that deformation. Also, notice that in the recent paper [7] the linear and nonlinear chiral supermultiplets were obtained by the reduction of the linear supermultiplet with four bosonic and four fermionic degrees of freedom [2, 14]. However, it is still unclear, how  $\lambda(z)$ -freedom could be explained in this picture.

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